STATISTICS IN TRANSITION new series, March 2026
Vol. XX, No. X, pp. xx–XX, https://doi.org/10.59139/stattrans-2026-xxx
Received – xx.xx.xxxx; accepted – xx.xx.xxxx

An extended odd log-logistic-lindley distribution with properties, applications and Bayesian estimation

Abbas Eftekharian[1], Morad Alizadeh[2], Vahid Ranjbar [3], Omid Kharazmi [4], Gholamhossein Hamedani [5]

Abstract

This paper introduces a four-parameter extended odd log-logistic-Lindley distribution from which moments, hazard, and quantile functions are then obtained. The statistical properties of this distribution show the high flexibility of the proposed distribution. The maximum likelihood and least-squares estimators of the extended odd log-logistic-Lindley parameters are studied. Moreover, a simulation study is carried out for evaluating the performance of the estimation methods, and the usefulness of the new distribution is illustrated using two real data sets. Finally, Bayesian analysis and efficiency of Gibbs sampling are provided on the basis of two real data sets.

Key words: Bayesian estimation, Gibbs sampling, Lindley distribution, moment, odd log-logistic, simulation.

1. Introduction

Modelling and analysing real lifetime data are widely used in many applied fields such as finance, reliability, engineering, medicine. In practice, researchers dealt with different types of survival data and they proposed various lifetime models for modelling such data. The statistical analysis depends on the procedure used by the researcher and the generated family of distributions. Recently, new families of distributions have been introduced in the literature that could considerably help to analyse complex real data. However, it is necessary to find more efficient statistical models; since there are many real data sets in practice that need to be investigated with statistical models that are more flexible. Therefore, the researchers have had many attempts to extend distributions theory by adding new shape parameters to different families of distribution to introduce new families. In particular, some extended distributions demonstrate high flexibility in hazard rate function (hrf) such

[1]University of Hormozgan, Iran. E-mail: ab2005eftekharian@gmail.com.
ORCID: https://orcid.org/0000-0002-5343-8597.
[2]Persian Gulf University, Iran. E-mail: m.alizadeh@pgu.ac.ir.
ORCID: https://orcid.org/0000-0001-6638-2185.
[3]Corresponding author. Golestan University, Iran. E-mail: vahidranjbar@gmail.com.
ORCID: https://orcid.org/0000-0003-3743-0330.
[4]Vali-e-Asr University of Rafsanjan, Iran. E-mail: omidkharazmi14@gmail.com.
ORCID: 0000-0003-4176-9708.
[5]Marquette University, United States, E-mail: gholamhoss.hamedani@marquette.edu.
ORCID: https://orcid.org/0000-0003-3216-0511.

as increasing, decreasing and bathtub shapes even though the baseline hazard rate function may not have these shapes.

Most of the new generators of G family can be obtained using T-X class, which is proposed by Alzaatreh et al. (2013). For example, Kumaraswamy generated, odd log-logistic-G, Exponentiated-G (Exp-G), gamma generated, proportional odds and generalized beta generated. Recently, the extended exponentiated-G (EE-G) family was defined by Alizadeh et al. (2018a).

In this paper, we introduce a new generator of G family using T-X class, which is called the extended odd log-logistic-G (EOLL-G) family, and study some of its mathematical properties. The main idea of the EOLL-G family is based on a contribution presented by Gleaton and Lynch (2010). They introduced an extended generalized log-logistic family for lifetime distribution. Following their idea and using T-X class the cumulative distribution function (cdf) of the EOLL-G family with parameters $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ as three additional shape parameters is defined by

$$F(x;\xi) = \int_0^{\frac{G(x;\theta)^\alpha}{[1-G(x;\theta)]^\beta}} \frac{\gamma}{(1+\gamma t)^2} dt = \frac{G(x;\theta)^\alpha}{G(x;\theta)^\alpha + \gamma[1-G(x;\theta)]^\beta}, \quad (1)$$

where $G(x;\theta)$ is the baseline cdf with the parameter vector θ and $\xi = (\alpha,\beta,\gamma,\theta)$.

It is clear that in the special case, the EOLL-G family reduces to EE-G family when $\beta = 1$. For $\alpha = \gamma = 1$, it transforms into Marshal-Olkin family. If $\beta = 1$ and $\alpha = \gamma$, then it reduces to Exp-G family. By considering $\alpha = \beta = \gamma = 1$, we obtain the baseline distribution G.

Gleaton and Lynch (2010) showed that the extended generalized log-logistic family has appropriate performance for lifetime data. Therefore, we can use the EOLL-G family for lifetime data by choosing a lifetime distribution as $G(\cdot)$ in (1). Although, there are several lifetime distributions that we can use, which is due to the fact that the proposed family has three parameters, it is better to select a lifetime distribution with only one parameter, for example, exponential or Lindley. It should be noted that hrf of the exponential is constant while the hrf of the Lindley distribution has different shapes as increasing, decreasing, unimodal and bathtub. Moreover, the Lindley distribution is a well-known distribution that is employed widely in different fields such as lifetime and reliability, medical, finance, engineering and insurance. These reasons motivate the use of this distribution for modeling real lifetime data. Therefore, we consider the Lindley distribution as the baseline distribution in this paper.

The Lindley distribution was originally proposed by Lindley (1958) in the Bayesian statistical context. Some properties of this distribution such as moments, failure rate function, characteristic function, mean residual life function, mean deviations, Lorenz curve, stochastic ordering, entropies, asymptotic distribution of the extreme order statistics have been studied by Ghitany et al. (2008). The cdf of the Lindley distribution with scale parameter $\lambda > 0$ is

$$G(x;\lambda) = 1 - \left(1 + \frac{\lambda x}{1+\lambda}\right) \mathrm{e}^{-\lambda x}, \; x > 0, \quad (2)$$

and its corresponding probability density function (pdf) is given by

$$g(x;\lambda)=\frac{\lambda^2}{1+\lambda}(1+x)\mathrm{e}^{-\lambda x}. \tag{3}$$

Many authors have published various extensions of the Lindley distribution recently. For example, a three-parameter generalization of the Lindley distribution proposed by Zakerzadeh and Dolati (2009), Nadarajah et al. (2011) defined a generalized Lindley distribution, a new generalized Lindley distribution based on the weighted mixture of two gamma distributions was studied by Abouammoh et al. (2015), Asgharzadeh et al. (2016, 2018) introduced a weighted Lindley distribution and Weibull Lindley distribution, respectively, and Alizadeh et al. (2017a,b,2018b) proposed several generalizations of the Lindley distribution based on the odd log-logistic model. Given the vast amount of papers published recently, we can only mention a few of the most recent contributions: Gomes-Silva et al. (2017), Afify et al. (2019), Alizadeh et al. (2019) and Alizadeh et al. (2025).

In the present paper, we introduce a new generalization of the Lindley distribution using the EOLL-G family. To this end, it is enough to choose the Lindley distribution as the baseline $G(x;\theta)$ in (1). By substituting (2) in (1), we get

$$F(x;\alpha,\beta,\gamma,\lambda)=\frac{\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha}}{\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha}+\gamma\left[\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\beta}},\quad x\geq 0, \tag{4}$$

and its corresponding pdf is given by

$$f(x;\alpha,\beta,\gamma,\lambda)=\frac{\gamma\lambda^2(1+x)\left(1+\frac{\lambda}{1+\lambda}x\right)^{\beta-1}\mathrm{e}^{-\beta\lambda x}\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha-1}}{(1+\lambda)\left\{\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha}+\gamma\left[\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\beta}\right\}^{2}}$$
$$\times\left\{\alpha+(\beta-\alpha)\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]\right\}. \tag{5}$$

A random variable X with pdf (5) has extended odd log-logistic-Lindley (EOLL-L) distribution and is denoted by $X\sim$ EOLL-L$(\alpha,\beta,\gamma,\lambda)$. The EOLL-L distribution is more flexible than the Lindley distribution and allows for greater flexibility of the tails.

Special cases: Let $X\sim EOLL-L(\alpha,\beta,\gamma,\lambda)$.

- If $\alpha=\beta,\gamma=1$, then EOLL-L reduces to the Odd Log-Logistic Lindley (OLL-L) Ozel et al. (2017).
- For $\alpha=\beta$, EOLL-L coincides with OLL-Marshall- Olkin Lindley (OLL-MOL) Alizadeh et al. (2017b).
- If $\alpha=\beta=1$, then XEOLL-L reduces to Marshall- Olkin Lindley (MOL).

- By taking $\gamma = 1$, EOLL-L coincides with the new OLL-Lindley (NOLLL) Alizadeh et al. (2018b).

- For $\alpha = \beta = \gamma = 1$, EOLL-L is ordinary Lindley.

The different shapes of the pdf such as unimodal, symmetric, skewed, and monotonically decreasing are shown in Figure 1 (left plot). As seen, the density of the EOLL-L model can be right-skewed density with one peak and heavy tail to the right, right-skewed density without a peak and with heavy tail to the right, bimodal and unimodal density with different shapes.

The point that catches our attention in this graph is that, for gamma values greater than 1, the density curve is symmetrical, and for less than 1, it is skewed to the right, and also for larger beta values, the tails of the distribution will become heavier.

The rest of the paper is organized as follows. In Section 2, some mathematical properties of the EOLL-L distribution are obtained. Certain characterizations are presented in Section 3. The estimations of the unknown parameters based on different methods are investigated in Section 4. A simulation study is reported in Section 5. In Section 6, the performance and application of the EOLL-L distribution are evaluated using two real data sets. Bayesian inference and Gibbs sampling procedure for the considered data sets are investigated in Section 7. Finally, some conclusions are stated in Section 8.

2. Main properties

2.1. Hazard rate function

In reliability studies, the hrf is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Using equations (4) and (5) the hrf of the EOLL-L distribution takes the form

$$\begin{aligned} h(x;\alpha,\beta,\lambda) &= \left(\frac{\lambda^2(1+x)\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha-1}}{(1+\lambda+\lambda x)\left\{\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha}+\gamma\left[\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\beta}\right\}} \right) \\ &\times \left\{\alpha+(\beta-\alpha)\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]\right\}. \end{aligned} \quad (6)$$

Plots for the hrfs for selected parameter values are displayed in Figure 1(right plot). As seen in Figure 1, the hrf of the EOLL-L distribution has very flexible shapes such as increasing, decreasing, upside-down, bathtub and upside-down-bathtub. It is evident that the EOLL-L distribution is more flexible than the Lindley distribution, in other words, the additional parameters $\alpha > 0, \beta > 0$ allow for a high degree of flexibility of the EOLL-L distribution. This attractive flexibility implies that the hrf of the EOLL-L is useful for non-monotone empirical hazard behaviour, which is more likely observed in real life situations.

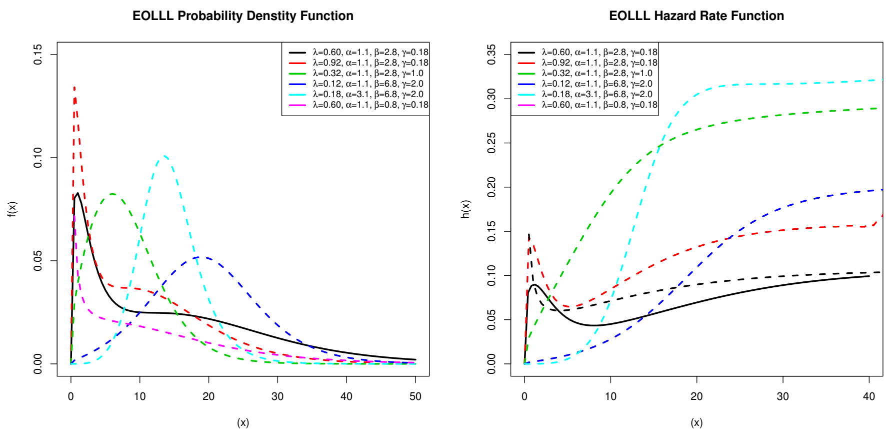

Figure 1. Plots of the density and hazard function for the EOLL-L distribution for selected parameter values

2.2. Quantile function

Quantile function is generally used to find representations in terms of lookup tables for key percentiles. Let X be an EOLL-L distributed random variable with parameters $\alpha, \beta, \lambda, \gamma$. The quantile function, $Q(p)$, defined by $F[Q(p)] = p$ is the root of the equation as

$$p = \frac{\left[1-\left(1+\frac{\lambda}{1+\lambda}\,Q(p)\right)\mathrm{e}^{-\lambda\,Q(p)}\right]^{\alpha}}{\left[1-\left(1+\frac{\lambda}{1+\lambda}\,Q(p)\right)\mathrm{e}^{-\lambda\,Q(p)}\right]^{\alpha}+\gamma\left[\left(1+\frac{\lambda}{1+\lambda}\,Q(p)\right)\mathrm{e}^{-\lambda\,Q(p)}\right]^{\beta}}. \tag{7}$$

A closed form of quantile function is available when $\alpha = \beta$. For this purpose, we define

$$[1+\lambda+\lambda\,Q(p)]\,\mathrm{e}^{-\lambda\,Q(p)} = \frac{(1+\lambda)\,(1-p)^{\frac{1}{\alpha}}}{(\gamma p)^{\frac{1}{\alpha}}+(1-p)^{\frac{1}{\alpha}}}, \tag{8}$$

for $0 < p < 1$. After some simple algebraic manipulation one can obtain

$$Q(p) = -1-\frac{1}{\lambda}-\frac{1}{\lambda}\,W_{-1}\left[\frac{-(1+\lambda)\,(1-p)^{\frac{1}{\alpha}}\,\mathrm{e}^{-1-\lambda}}{(\gamma p)^{\frac{1}{\alpha}}+(1-p)^{\frac{1}{\alpha}}}\right]. \tag{9}$$

where $W_{-1}[.]$ is the negative branch of the Lambert function (Corless et al. 1996). Note that the particular case of (9) for $\alpha = \beta = \gamma = 1$ is derived by Jodr (2010).

Now, we propose the following algorithm for generating random data from the EOLL-L distribution for the case $\alpha = \beta$.

Algorithm 1 (Inverse cdf)

- Generate $U_i \sim$ Uniform(0,1), $i = 1,\ldots,n$;
- Set

$$X_i = \left\{ -1 - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1} \left[\frac{-(1+\lambda)(1-U_i)^{\frac{1}{\alpha}} \mathrm{e}^{-1-\lambda}}{(\gamma U_i)^{\frac{1}{\alpha}} + (1-U_i)^{\frac{1}{\alpha}}} \right] \right\}, \quad i = 1,\ldots,n.$$

For $\alpha \neq \beta$, we applied the following algorithm for generating random data:

- Step 1. Generate random numbers u_i from $U \sim U(0,1)$ for $i = 1,\cdots,n$.
- Step 2. Select arbitrary values for parameters of EOLL-L distribution, i.e. α, β, γ and λ.
- Step 3. Solve numerically the non-linear equation

$$u_i = \frac{\left[1-(1+\frac{\lambda x_i}{1+\lambda})\mathrm{e}^{-\lambda \, \mathrm{x_i}}\right]^{\alpha}}{\left[1-(1+\frac{\lambda x_i}{1+\lambda})\mathrm{e}^{-\lambda \, \mathrm{x_i}}\right]^{\alpha} + \gamma\left[(1+\frac{\lambda x_i}{1+\lambda})\mathrm{e}^{-\lambda \, \mathrm{x_i}}\right]^{\beta}}, \tag{10}$$

and compute values of x_i for $i = 1,\cdots,n$.

2.3. Expansions for the density and cumulative distribution functions

In this subsection, two mixture representations of the pdf and cdf for EOLL-L are proposed. Despite the fact that the pdf and cdf of EOLL-L require mathematical functions that are widely available in modern statistical packages, frequently analytical and numerical derivations take advantage of power series representations for the pdf. Therefore, we use the concept of power series to calculate the useful expansions. Accordingly, the pdf of the EOLL-L distribution is given by

$$\left\{1-\left(1+\frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x}\right\}^{\alpha} = \sum_{k=0}^{\infty} a_k \left\{1-\left(1+\frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x}\right\}^{k}, \tag{11}$$

where $a_k = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}$ and

$$\left[1-\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\alpha} + \gamma\left[\left(1+\frac{\lambda}{1+\lambda}x\right)\mathrm{e}^{-\lambda x}\right]^{\beta} = \sum_{k=0}^{\infty} b_k \left\{1-\left(1+\frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x}\right\}^{k}, \tag{12}$$

where $b_k = a_k + \gamma(-1)^k\binom{\beta}{k}$. Then, we can write

$$F(x) = \frac{\left\{1-\left(1+\frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x}\right\}^{\alpha}}{\sum_{k=0}^{\infty} b_k \left\{1-\left(1+\frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x}\right\}^{k}} = \sum_{k=0}^{\infty} c_k \left\{1-\left(1+\frac{\lambda x}{1+\lambda}\right)\mathrm{e}^{-\lambda x}\right\}^{k+\alpha}. \tag{13}$$

where $c_0 = \frac{1}{b_0}$ and for $k \geq 1$,

$$c_k = -b_0^{-1} \sum_{r=1}^{k} b_r \, c_{k-r}. \tag{14}$$

Hence, the cdf of the EOLL-L distribution can be written as

$$F(x) \quad = \quad \sum_{k=0}^{\infty} c_k \, G_{k+\alpha}(x). \tag{15}$$

where $G_{k+\alpha}(x)$ denotes the cdf of the generalized Lindley (exponentiated Lindley) distribution with parameters λ and $k+\alpha$.

Moreover, by differentiating from (15), the pdf of X can be expressed as

$$f(x) \quad = \quad \sum_{k=0}^{\infty} c_k \, g_{k+\alpha}(x). \tag{16}$$

where $g_{k+\alpha}(x)$ is the pdf of the generalized Lindley distribution with parameters λ and $k+\alpha$. Several properties of the EOLL-L distribution can be available from the cdf and pdf expansions, given in (15) and (16), respectively.

2.4. Moments and moment generating function

Some of the most important features and characteristics of a distribution can be investigated through moments (e.g., central tendency, dispersion, skewness, and kurtosis). In what follows, we present ordinary moments and the moment generating function (mgf) of the EOLL-L distribution. To find the ordinary moments (μ_r'), we use the following equation, which is introduced by Nadarajah et al. (2011) as

$$A(a,b,c,\delta) = \int_0^{\infty} x^c (1+x) \left[1 - \left(1 + \frac{bx}{b+1}\right) \mathrm{e}^{-bx}\right]^{a-1} \mathrm{e}^{-\delta x} dx. \tag{17}$$

From (17), we have

$$A(a,b,c,\delta) = \sum_{l=0}^{\infty} \sum_{r=0}^{l} \sum_{s=0}^{r+1} \binom{a-1}{l} \binom{l}{r} \binom{r+1}{s} \frac{(-1)^l b^r \Gamma(s+c+1)}{(1+b)^l (bl+\delta)^{c+s+1}}. \tag{18}$$

Using equations (15) and (16), we get the ordinary moments of the EOLL-L distribution as

$$\mu_r' = E\left[X^r\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+\alpha) \, c_k \, A(k+\alpha, \lambda, r, \lambda). \tag{19}$$

We now provide a formula for the conditional moments of the EOLL-L distribution. To this end, we use the following equation, which is introduced by Nadarajah et al. (2011) as

$$L(a,b,c,\delta,t) = \int_t^{\infty} x^c (1+x) \left[1 - \left(1 + \frac{bx}{b+1}\right) \mathrm{e}^{-bx}\right] \mathrm{e}^{-\delta x} dx. \tag{20}$$

Using the generalized binomial expansion, we have

$$L(a,b,c,\delta,t) = \sum_{l=0}^{\infty} \sum_{r=0}^{l} \sum_{s=0}^{r+1} \binom{a-1}{l} \binom{l}{r} \binom{r+1}{s} \frac{(-1)^l b^r \Gamma(s+c+1, (bl+\delta)t)}{(1+b)^l (bl+\delta)^{c+s+1}}, \tag{21}$$

where

$$\Gamma(a,x)=\int_x^\infty t^{a-1}\,\mathrm{e}^{-t}\,dt, \tag{22}$$

denotes the incomplete gamma function. From equations (16) and (21), we obtain the conditional moments of the EOLL-L distribution as

$$\mu_r'(t)=E\left[X^r|X>t\right]=\frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+\alpha)\,c_k\,L(k+\alpha,\lambda,r,\lambda,t). \tag{23}$$

Moreover, the incomplete moments of the EOLL-L distribution can be obtained directly from (23).

Using (16) and (18), we can derive the mgf as follows:

$$M_X(t)=E\left[\mathrm{e}^{\mathrm{tX}}\right]=\frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+\alpha)\,c_k A(k+\alpha,\lambda,0,\lambda-t).$$

Remark 1 *The central moments (μ_n) and cumulants (κ_n) of X are easily calculated from (19) (e.g. see Arellano-Valle et al., 2017, and Contreras-Reyes et al., 2021) as*

$$\mu_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu_1'^k\,\mu_{n-k}' \qquad \text{and} \qquad \kappa_n=\mu_n'-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\kappa_k\,\mu_{n-k}',$$

respectively, where $\kappa_1=\mu_1'$. Thus, $\kappa_2=\mu_2'-\mu_1'^2$, $\kappa_3=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3$, etc.

From the ordinary moments and using (19) the mean, variance, skewness and kurtosis are calculated for different values of parameters in Table 1. From Table 1 it can be seen that skewness and kurtosis are very sensitive to changes in the shape parameters so, the importance of the proposed distribution can be concluded.

Table 1. Moments, skewness, and kurtosis of EOLL-L distribution for some parameters values

α	γ	β	λ	μ_1'	μ_2'	μ_3'	μ_4'	*Skewness*	*Kurtosis*
0.5	0.5	1.0	0.5	1.903213	9.702343	72.93571	707.2085	2.089507	53.199973
0.5	0.5	1.0	2.0	0.371702	0.428410	0.756338	1.754821	2.438598	3.1978419
0.5	0.5	2.0	0.5	1.248766	3.631016	15.00114	78.75589	1.775172	14.724114
0.5	0.5	2.0	2.0	0.225271	0.136348	0.124155	0.148577	2.190991	0.8235115
0.5	2.0	1.0	0.5	4.006446	27.10376	236.9726	2507.292	1.083841	49.493653
0.5	2.0	2.0	2.0	0.473993	0.383384	0.407392	0.531980	1.189489	0.7872935
3.0	1.5	1.0	0.5	4.985129	31.22020	242.6686	2311.769	1.464291	43.219819
1.5	0.5	2.0	1.5	0.535931	0.438362	0.494825	0.729813	1.666026	1.1710862
2.0	2.5	1.5	3.0	0.530031	0.370096	0.323693	0.345280	1.240077	0.5167241
2.0	0.5	0.5	1.0	1.409837	3.076611	9.717392	41.62731	2.03217	10.713407

3. Estimation

Point estimation is the first step of statistical inference on the unknown parameters of the underlying population. In order to find point estimations, there are different methods such as maximum likelihood estimation (MLE), least square and moment method. In the present paper, we obtain the maximum likelihood, least square and weighted least-square estimations for the parameters of the EOLL-L distribution.

3.1. Maximum likelihood estimation

Let $X_1,\ldots,X_n$ be a random sample of size n from the EOLL-L$(\alpha,\beta,\gamma,\lambda)$ distribution. The log-likelihood function for the vector of parameters $\theta=(\alpha,\beta,\gamma,\lambda)^T$ can be written as

$$\begin{aligned} l(\theta) &= n\log\left(\frac{\gamma\lambda^2}{1+\lambda}\right)+\sum_{i=1}^{n}\log(1+x_i)+(\beta-1)\sum_{i=1}^{n}\log(1+\frac{\lambda x_i}{1+\lambda})-\beta\lambda\sum_{i=1}^{n}x_i \\ &+ (\alpha-1)\sum_{i=1}^{n}\log(q_i)+\sum_{i=1}^{n}\log\left[\alpha+(\beta-\alpha)q_i\right]-2\sum_{i=1}^{n}\log\left[q_i^{\alpha}+\gamma(1-q_i)^{\beta}\right] \quad (24)\end{aligned}$$

where $q_i=1-(1+\frac{\lambda}{1+\lambda}x_i)\mathrm{e}^{-\lambda x_i}$ is a transformed observation. The log-likelihood can be maximized by differentiating (24) and solving the nonlinear likelihood equations. The components of the score vector $U(\theta)$ are given by

$$\begin{aligned} U_\lambda(\theta) &= \frac{2n}{\lambda}-\frac{n}{1+\lambda}-\beta\sum_{i=1}^{n}x_i+(\beta-1)\sum_{i=1}^{n}\frac{x_i}{(1+\lambda)(1+\lambda+\lambda x_i)} \\ &+ (\alpha-1)\sum_{i=1}^{n}\frac{q_i^{(\lambda)}}{q_i}+(\beta-\alpha)\sum_{i=1}^{n}\frac{q_i^{(\lambda)}}{\alpha+(\beta-\alpha)q_i} \\ &-2\sum_{i=1}^{n}q_i^{(\lambda)}\ \frac{\alpha q_i^{\alpha-1}-\gamma\beta(1-q_i)^{\beta-1}}{q_i^{\alpha}+\gamma(1-q_i)^{\beta}}, \\ U_\alpha(\theta) &= \sum_{i=1}^{n}\log(q_i)+\sum_{i=1}^{n}\frac{1-q_i}{\alpha+(\beta-\alpha)q_i}-2\sum_{i=1}^{n}\frac{q_i^{\alpha}\log(q_i)}{q_i^{\alpha}+\gamma(1-q_i)^{\beta}}, \\ U_\gamma(\theta) &= \frac{n}{\gamma}-2\sum_{i=1}^{n}\frac{(1-q_i)^{\beta}}{q_i^{\alpha}+\gamma(1-q_i)^{\beta}},\end{aligned}$$

$$\begin{aligned} U_\beta(\theta) &= \sum_{i=1}^{n}\log(1+\frac{\lambda x_i}{1+\lambda})-\lambda\sum_{i=1}^{n}x_i \\ &+ \sum_{i=1}^{n}\frac{q_i}{\alpha+(\beta-\alpha)q_i}-2\sum_{i=1}^{n}\frac{(1-q_i)^{\beta}\log(1-q_i)}{q_i^{\alpha}+\gamma(1-q_i)^{\beta}}.\end{aligned}$$

To construct a confidence interval and find test statistic for testing hypothesis on the parameters, the 4×4 observed information matrix $J=J(\theta)$ is required.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ is $N_4(0,I(\theta)^{-1})$, where $I(\theta)$ is the expected information matrix. In practice, we can replace $I(\theta)$ by the observed information matrix evaluated at $\hat{\theta}$ (say $J(\hat{\theta})$). We can construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard and survival functions based on the multivariate normal $N_4(0,J(\hat{\theta})^{-1})$ distribution.

3.2. Ordinary and weighted least-square estimators

Let $x_{(1)}, x_{(2)}, x_{(3)}, \cdots, x_{(n)}$ denote the ordered sample of the random observations of size n from the EOLL-L distribution. By minimizing the following equation

$$\ell(\theta)=\sum_{i=1}^{n}\left(\frac{\left[1-\left(1+\frac{\lambda}{1+\lambda}x_{(i)}\right)e^{-\lambda x_{(i)}}\right]^{\alpha}}{\left[1-\left(1+\frac{\lambda}{1+\lambda}x_{(i)}\right)e^{-\lambda x_{(i)}}\right]^{\alpha}+\gamma\left[\left(1+\frac{\lambda}{1+\lambda}x_{(i)}\right)e^{-\lambda x_{(i)}}\right]^{\beta}}-\frac{i}{n+1}\right)^{2}, \quad (25)$$

the least-square estimations (LSEs) of the EOLL-L distribution can be computed. Moreover, the weighted least square estimators (WLSEs) of the EOLL-L distribution can be derived by minimizing the following equation

$$\ell(\theta)=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(\frac{\left[1-\left(1+\frac{\lambda}{1+\lambda}x_{(i)}\right)e^{-\lambda x_{(i)}}\right]^{\alpha}}{\left[1-\left(1+\frac{\lambda}{1+\lambda}x_{(i)}\right)e^{-\lambda x_{(i)}}\right]^{\alpha}+\gamma\left[\left(1+\frac{\lambda}{1+\lambda}x_{(i)}\right)e^{-\lambda x_{(i)}}\right]^{\beta}}-\frac{i}{n+1}\right)^{2}. \quad (26)$$

One can use the **optim** function in R software to minimize the (25) and (26). The partial derivatives of (25) and (26) with respect to α, β, γ and λ can be obtained from the authors upon request.

4. Simulation

In this section, a simulation study on the model parameters is investigated. The MLE, LSE, and WLSE methods are used for estimating the unknown parameters of the EOLL-L distribution and the performance of the methods are compared. The simulation procedure has been performed according to the following steps:

1. Set the sample size n and the vector of parameters $\theta=(\alpha,\beta,\gamma,\lambda)$.
2. Generate random observations from the $EOLL-L(\alpha,\beta,\gamma,\lambda)$ distribution with size n using Algorithm 1 in subsection 2.2.
3. Apply the generated random observations in Step 2 and estimate $\hat{\theta}$ by means of MLE, LSE and WLSE methods.
4. Repeat Steps 2 and 3 for N times.
5. Compute the mean relative estimates (MREs) and mean square errors (MSEs) using $\hat{\theta}$ and θ on the basis of the following equations:

$$MRE=\sum_{j=1}^{N}\frac{\hat{\theta}_{i,j}/\theta_{i}}{N}, \qquad MSE=\sum_{j=1}^{N}\frac{\left(\hat{\theta}_{i,j}-\theta_{i}\right)^{2}}{N},$$

where $\hat{\theta}_{i,j}$ for $i=1,..,4$ and $j=1,...,N$, is the estimation of ith element of parameter vector in jth iteration. The simulation results are obtained with R software. The chosen parameters of the simulation study are $\theta=(\alpha=0.5,\beta=2,\gamma=1.5,\lambda=2.5)$, $N=1000$ and

$n = (50, 55, 60, ..., 500)$. We expect that MREs are closer to one when the MSEs are near zero. Figure 2 represents estimated MSEs and MREs based on the MLE, LSE and WLSE methods. As expected, MSEs and MREs of all estimates tend to zero and one for large n, respectively. Furthermore, it is deduced generally that the LSE method has better performance than the MLE method as well as the WLSE method to estimate EOLL-L parameters based on both MSE and MRE criteria even for the small sample size.

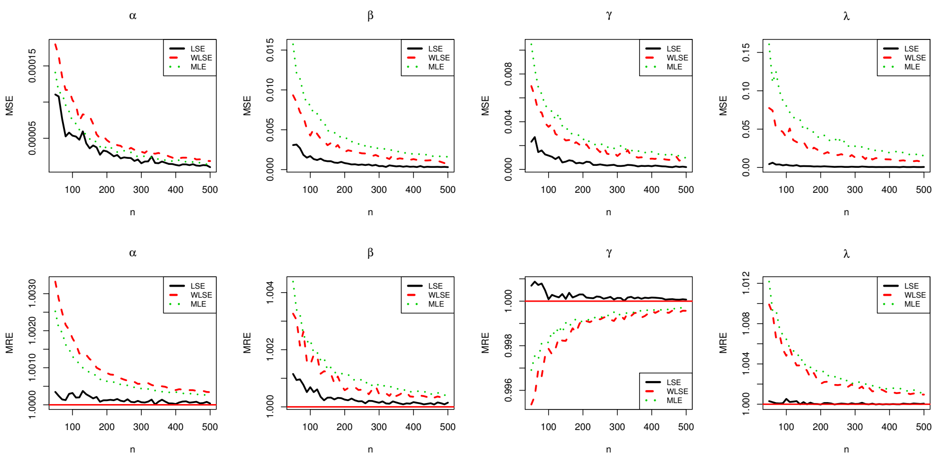

Figure 2. The behavior of MSEs and MREs of MLE, LSE and WLSE methods for different values of sample size.

5. Applications

In this section, we illustrate the fitting performance of the EOLL-L distribution using a real data sets. To evaluate the performance of the EOLL-L distribution, we recall a few extended of Lindley distribution such as: Power Lindley distribution, $PL(\beta,\lambda)$ (Ghitany et al. (2013)), Generalized Lindley, $GL(\alpha,\lambda)$, (Nadarajah et al. (2011)), Beta Lindley, $BL(\alpha,\beta,\lambda)$, (Merovci and Sharma (2014)), Exponentiated power Lindley distribution, $EPL(\alpha,\beta,\lambda)$, (Ashour and Eltehiwy (2015)), Odd log-logistic power Lindley distribution, $OLL-PL(\alpha,\beta,\lambda)$, (Alizadeh et al. (2017a)), Kumaraswamy Power Lindley, $Kw(\alpha,\beta,\gamma,\lambda)$, (Oluyede et al. (2016)), Odd Burr- Lindley, $OBu-L(\alpha,\beta,\lambda)$ (Altun et al. (2017)), Extended generalized Lindley, $EGL(\alpha,\gamma,\lambda)$, (Ranjbar et al. (2019)), Marshal-Olkin Lindley, $MOL(\gamma,\lambda)$, (Marshall and Olkin (1997)), New odd-log logistic Lindley, $NOLLL(\alpha,\beta,\lambda)$, (Alizadeh et al. (2018b)), Odd-log logistic Marshal-Olkin Lindley, $OLL-MOL(\alpha,\gamma,\lambda)$, (Alizadeh et al. (2017b)).

To compare the EOLL-L distribution with the above-mentioned distributions we consider several well-known criteria such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer Von Mises (W^*) and Anderson-Darling (A^*) statistics. In addition, Kolmogorov-Smirnov (K-S) statistic with its corresponding p-value and minimum value of minus log-likelihood function (-Log(L)) are investigated for all distributions.

Furthermore, the likelihood ratio (LR) tests apply for evaluating the EOLL-L distribution with its sub-models. For example, the test of $H_0 : \beta = 1$ against $H_1 : \beta \neq 1$ is equivalent to comparing the EOLL-L with EGL, and the LR test statistic is given by

$$LR = 2\left[l(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\lambda}) - l(\hat{\alpha^*},1,\hat{\gamma^*},\hat{\lambda^*})\right],$$

where $\hat{\alpha^*}, \hat{\gamma^*}$ and $\hat{\lambda^*}$ are the ML estimators under H_0 of α, γ and λ, respectively. It should be highlighted that the initial values of the parameters are quite important to obtain the correct MLEs of parameters. To avoid the local minima problem, we first obtain the parameter estimation of the Lindley distribution. Then, the estimated parameter of the Lindley distribution is used as the initial value of the parameter in all the mentioned extended of the Lindley distribution as well as the EOLL-L distribution. This approach is quite useful to obtain correct parameter estimates of extended distributions.

The data are the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984 rounded to one decimal place. These data were analyzed by Akinsete et al. (2008). Throughout this subsection, we present the obtained results using the exceedances of the flood peaks data set.

Table 2. The exceedances of flood peaks data set

1.70	2.20	14.4	1.10	0.40	20.6	5.30	0.70	1.90	13.0	12.0	9.30	1.40
18.7	8.50	25.5	11.6	14.1	22.1	1.10	2.50	14.4	1.70	37.6	0.60	2.20
39.0	0.30	15.0	11.0	7.30	22.9	1.70	0.10	1.10	0.60	9.00	1.70	7.00
20.1	0.40	2.80	14.1	9.90	10.4	10.7	30.0	3.60	5.60	30.8	13.3	4.20
25.5	3.40	11.9	21.5	27.6	36.4	2.70	64.0	1.50	2.50	27.4	1.00	27.1
20.2	16.8	5.30	9.70	27.5	2.50	27.0						

The ML estimates and the goodness-of-fit test statistics are presented in Tables 3 and 4, respectively. From Table 4, the smallest values of AIC, A^*, W^* and $-l$ statistics and the largest p-value belong to the EOLL-L distribution. Although, the BIC of OLLPL is less than that of EOLLL, in general, the EOLL-L distribution outperforms the other competitive considered distributions on the basis of the criteria. The values of LR test statistics and

Table 3. The ML estimates and their standard errors (in parentheses) for first data set

Model	α	β	γ	λ
Lindley(λ)	–	–	–	0.153 (0.0128)
GL(α,λ)	0.508 (0.0767)	–	–	0.104 (0.01491)
PL(β,λ)	–	0.700 (0.0570)	–	0.338 (0.0559)
BL(α,β,λ)	0.555 (0.0983)	0.274 (0.2397)	–	0.333 (0.2723)
EPL(α,β,λ)	0.730 (0.2351)	0.915 (0.5956)	–	0.300 (0.2791)
OLLPL(α,β,λ)	0.183 (0.0222)	–	–	0.612 (0.0660)
KwL($\alpha,\beta,\gamma,\lambda$)	1.675 (2.4335)	0.453 (0.4323)	7.563 (11.7366)	0.279 (0.5225)
OBuL(α,β,λ)	24.91 (25.654)	0.024 (0.0326)	–	0.984 (0.1496)
EGL(α,γ,λ)	0.618 (0.1018)	–	2.770 (1.7047)	0.169 (0.0288)
MOL(γ,λ)	–	–	0.215 (0.1276)	0.090 (0.0246)
NOLLL(α,β,λ)	1.1735(0.1917)	–	0.171 (0.0238)	0.547 (0.0262)
OLLMOL(α,γ,λ)	0.6165(0.0880)	–	0.965 (0.4366)	0.180 (0.0470)
EOLLL($\alpha,\beta,\gamma,\lambda$)	1.113 (0.2132)	1.775 (0.4509)	0.176 (0.0244)	0.618 (0.0026)

Table 4. Goodness-of-fit test statistics for the data set

Model	AIC	BIC	p-value	W^*	A^*	$-l$
Lindley(λ)	530.423	532.700	0.001	0.139	0.852	264.211
GL(α,λ)	509.349	513.902	0.276	0.132	0.822	252.674
PL(β,λ)	508.443	512.996	0.405	0.123	0.766	252.103
BL(α,β,λ)	510.206	517.036	0.297	0.150	0.866	252.221
EPL(α,β,λ)	510.425	517.255	0.395	0.147	0.854	252.212
OLLPL(α,β,λ)	506.029	510.582	0.501	0.100	0.621	251.015
KwL($\alpha,\beta,\gamma,\lambda$)	512.221	521.328	0.371	0.152	0.866	252.110
OBuL(α,β,λ)	511.212	520.319	0.401	0.140	0.799	251.606
EGL(α,γ,λ)	508.931	515.761	0.174	0.101	0.662	251.465
MOL(γ,λ)	522.570	527.124	0.024	0.214	1.208	259.285
NOLLL(α,β,λ)	506.505	513.335	0.035	0.095	0.517	250.252
OLLMOL(α,γ,λ)	508.023	514.853	0.517	0.101	0.623	251.011
EOLLL($\alpha,\beta,\gamma,\lambda$)	502.327	511.433	0.958	0.041	0.249	247.163

Table 5. The LR test results for the data set

	Hypotheses	LR	p-value
EOLL-L versus Lindley	$H_0: \alpha=\beta=\gamma=1$	34.0966	< 0.0001
EOLL-L versus OLL-L	$H_0: \alpha=\beta,\ \gamma=1$	7.7022	0.0212
EOLL-L versus MOL	$H_0: \alpha=\beta=1$	24.2439	< 0.0001
EOLL-L versus NOLLL	$H_0: \gamma=1$	6.1782	0.01293
EOLL-L versus OLL-MOL	$H_0: \alpha=\beta$	7.6962	0.00553

their corresponding p-values are exhibited in Table 5. From Table 5, we observe that the computed p-values are too small so we reject all the null hypotheses and conclude that the EOLL-L fits the data set better than the considered sub-models according to the LR criterion.

We also plotted the fitted pdfs, cdfs and P-P plots of the considered models for the sake of visual comparison, in Figures 4 and 5, respectively. Figure 4 suggests that the EOLL-L fits the skewed data very well. Figures 5 shows that the plotted points for the EOLL-L distribution best capture the diagonal line in the probability plots. Therefore, the EOLL-L distribution can be considered as an appropriate model for fitting the first data set.

6. Bayesian estimation

The Bayesian inference procedure has been taken into consideration by many statistical researchers, especially researchers in the field of survival analysis and reliability engineering. In this section, a complete sample data is analysed through the Bayesian point of view. We assume that the parameters α, β, γ and λ of the $EOLL-L$ distribution have independent prior distributions as

$$\alpha \sim Gamma(a,b), \gamma \sim Gamma(c,d), \lambda \sim Gamma(e,f), \beta \sim Gamma(g,h)$$

where a,b,c,d,e, f, g and h are positive. Hence, the joint prior density function is formulated as follows:

$$\pi(\alpha,\beta,\gamma,\lambda) = \frac{b^a d^c f^e h^g}{\Gamma(a)\Gamma(c)\Gamma(e)\Gamma(g)} \alpha^{a-1} \beta^{h-1} \gamma^{c-1} \lambda^{e-1} e^{-(b\alpha+h\beta+d\gamma+f\lambda)}. \quad (27)$$

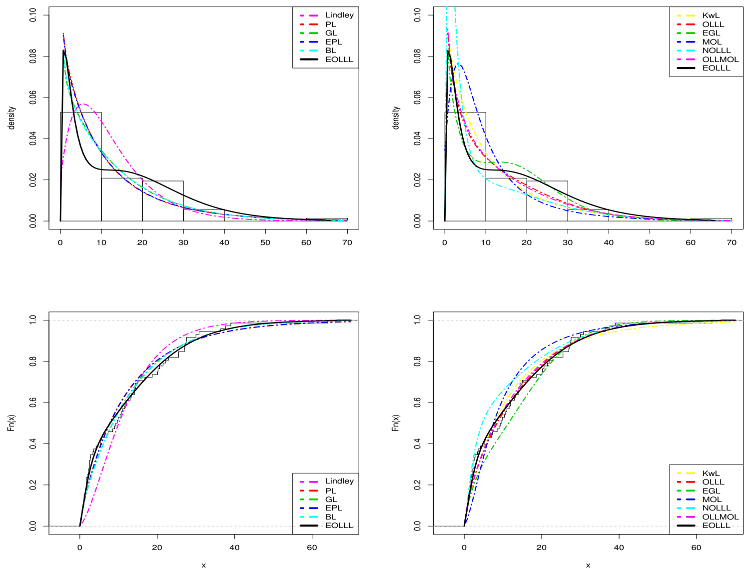

Figure 3. Fitted pdfs and cdfs of the distributions for the data set

In the Bayesian estimation, according to which we do not know the actual value of the parameter, we may be adversely affected by loss when we choose an estimator. This loss can be measured by a function of the parameter and the corresponding estimator.

Five well-known loss functions and associated Bayesian estimators and corresponding posterior risks are presented in Table 6. For more details, the reader can refer to Calabria and Pulcini (1996). Next, we provide the posterior probability distributions for a complete data set. Let us we define the function φ as

Table 6. Bayes estimator and posterior risk under different loss functions

Loss function	Bayes estimator	Posterior risk
$L_1 = SELF = (\theta - d)^2$	$E(\theta\|x)$	$Var(\theta\|x)$
$L_2 = WSELF = \frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1}\|x))^{-1}$	$E(\theta\|x) - (E(\theta^{-1}\|x))^{-1}$
$L_3 = MSELF = \left(1 - \frac{d}{\theta}\right)^2$	$\frac{E(\theta^{-1}\|x)}{E(\theta^{-2}\|x)}$	$1 - \frac{E(\theta^{-1}\|x)^2}{E(\theta^{-2}\|x)}$
$L_4 = PLF = \frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2\|x)}$	$2\left(\sqrt{E(\theta^2\|x)} - E(\theta\|x)\right)$
$L_5 = KLF = \left(\sqrt{\frac{d}{\theta}} - \sqrt{\frac{\theta}{d}}\right)$	$\sqrt{\frac{E(\theta\|x)}{E(\theta^{-1}\|x)}}$	$2\left(\sqrt{E(\theta\|x)E(\theta^{-1}\|x)} - 1\right)$

$$\varphi(\alpha,\beta,\gamma,\lambda) = \alpha^{a-1}\beta^{h-1}\gamma^{c-1}\lambda^{e-1}e^{-(b\alpha+h\beta+d\gamma+f\lambda)},\ \alpha>0,\ \beta>0,\ \gamma>0,\ \lambda>0.$$

The joint posterior distribution in terms of a given likelihood function $L(data)$ and joint prior distribution $\pi(\alpha,\beta,\gamma,\lambda)$ is defined as

$$\pi^*(\alpha,\beta,\gamma,\lambda|data) \propto \pi(\alpha,\beta,\gamma,\lambda)L(data). \quad (28)$$

Table 7. Bayesian estimates $\widehat{\theta}$ and their posterior risks $r_{\widehat{\theta}}$ of the parameters under different loss functions based on the flood peaks data.

Data	Flood peaks			
Bayesian estimation				
Loss function	$\widehat{\alpha}$ $(r_{\widehat{\alpha}})$	$\widehat{\beta}$ $(r_{\widehat{\beta}})$	$\widehat{\gamma}$ $(r_{\widehat{\gamma}})$	$\widehat{\lambda}$ $(r_{\widehat{\lambda}})$
SELF	1.5331 (0.0863)	0.1852 (0.0012)	1.3004 (0.0386)	0.5993 (0.0076)
WSELF	1.4771 (0.0561)	0.1791 (0.0061)	1.2702 (0.0302)	0.5858 (0.0135)
MSELF	1.4217 (0.0375)	0.1735 (0.0312)	1.2396 (0.0241)	0.6056 (0.0247)
PLF	1.5610 (0.0557)	0.1886 (0.0067)	1.3152 (0.0295)	0.6056 (0.0126)
KLF	1.5049 (0.0376)	0.1821 (0.0338)	1.2852 (0.0236)	0.5925 (0.0229)

Table 8. Credible and *HPD* intervals of the parameters α, β, γ and λ for the flood peaks.

	Credible interval	HPD interval
α	(1.329, 1.737)	(0.957, 2.068)
β	(1.161, 1.429)	(0.949, 1.703)
γ	(0.160, 0.205)	(0.124, 0.254)
λ	(0.542, 0.662)	(0.428, 0.760)

Hence, we get joint posterior density of parameters α, β, γ and λ for complete sample data by combining the likelihood function and joint prior density (27). Therefore, the joint posterior density function is given by

$$\pi^*(\alpha,\beta,\gamma,\lambda|\underline{x}) \quad = \quad K\varphi(\alpha,\beta,\gamma,\lambda)L(\underline{x},\xi) \tag{29}$$

where

$$L(\underline{x};\xi)=\prod_{i=1}^{n}\frac{\gamma\lambda^2(1+x_i)\mathrm{e}^{-\beta\lambda\,\mathrm{x_i}}\left[1-(1+\frac{\lambda \mathrm{x_i}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x_i}}\right]^{\alpha-1}\left\{\alpha+(\beta-\alpha)\left[1-(1+\frac{\lambda \mathrm{x_i}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x_i}}\right]\right\}}{(1+\lambda)\left\{\left[1-(1+\frac{\lambda\,x_i}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x_i}}\right]^{\alpha}+\gamma\left[(1-(1+\frac{\lambda\,x_i}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x_i}})^{\beta}\right]\right\}^2}. \tag{30}$$

and K is given as

$$K^{-1} \quad = \quad \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \varphi(\alpha,\beta,\gamma,\lambda)L(\underline{x},\xi)d\alpha d\beta d\gamma d\lambda.$$

It is clear from equation (29) that there is no closed form for the Bayesian estimators under the five loss functions described in Table 6, so we suggest using an *MCMC* procedure based on 10000 replicates to compute Bayesian estimators. The corresponding Bayesian point and interval estimation and posterior risk are provided in Tables 7 and 8 for the flood peaks data set. Table 8 provides 95% credible and *HPD* intervals for each parameter of the $EOLL-L$ distribution. The posterior samples are extracted using Gibbs sampling technique. Moreover, we provide the posterior summary plots in Figure 4. These plots confirm that the convergence of Gibbs sampling process occurred.

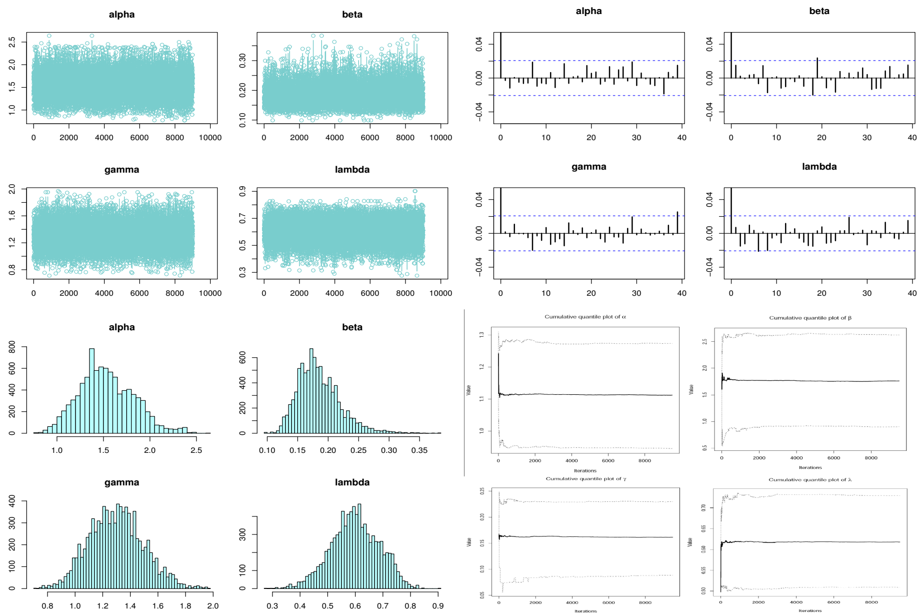

Figure 4. Plots of Bayesian analysis and performance of Gibbs sampling for the flood peaks data set.

7. Conclusion

In this paper, a new distribution which is called extended odd log-logistic-Lindley (EOLL-L) distribution was introduced. The statistical properties of the EOLL-L distribution including the hazard function, quantile function, moments, incomplete moments and generating functions and maximum likelihood estimation for the model parameters were given. Simulation studies were conducted to examine the performance of this distribution. We also presented applications of this new distribution for two real-life data sets in order to illustrate the usefulness of the distribution. Finally, the Bayesian estimation and Gibbs sampling procedure for the considered data sets were discussed.

References

Abouammoh, A., Alshangiti, A. M. and Ragab, I., (2015). A new generalized Lindley distribution. *Journal of Statistical Computation and Simulation*, 85(18), pp. 3662–3678.

Afify, A. Z., Cordeiro, G. M., Maed, M. E., Alizadeh, M., Al-Mofleh, H. and Nofal, Z. M., (2019). The generalized odd lindley-g family: properties and applications. *Anais da Academia Brasileira de Ciencias*, 91(3).

Akinsete, A., Famoye, F. and Lee, C., (2008). The beta-pareto distribution. *Statistics*, 42(6), pp. 547–563.

Alizadeh, M., Afshari, M., Hosseini, B. and Ramires, T. G., (2018a). Extended exp-g family of distributions: Properties and applications. *Communication in Statistics-Simulation and Computation*, accepted.

Alizadeh, M., Afify, A. Z., Eliwa, M. and Ali, S., (2019). The odd log-logistic lindley-g family of distributions:properties, bayesian and non-bayesian estimation with applications. *Computational Statistics*, pp. 1–28.

Alizadeh, M., Altun, E., Ozel, G., Afshari, M., and Eftekharian, A., (2018b). A new odd log-logistic lindley distribution with properties and applications. *Sankhya A*, 81(2), pp. 323–346.

Alizadeh, M., K MirMostafaee, S., Altun, E., Ozel, G. and Khan Ahmadi, M., (2017a). The odd log-logistic marshall-olkin power lindley distribution: Properties and applications. *Journal of Statistics and Management Systems*, 20(6), pp. 1065–1093.

Alizadeh, M., Ozel, G., Altun, E., Abdi, M. and Hamedani, G. (2017b). The odd log-logistic marshall-olkin lindley model for lifetime data. *Journal of Statistical Theory and Applications*, 16(3), pp. 382–400.

Alizadeh, M., Afshari, M., Cordeiro, G. M., Ramaki, Z., Contreras-Reyes, J. E., Dirnik, F. and Yousof, H. M., (2025). A Weighted Lindley Claims Model with Applications to Extreme Historical Insurance Claims. *Stats*, 8(1), p. 8.

Altun, G., Alizadeh, M., Altun, E. and Ozel, G., (2017). Odd burr lindley distribution with properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 46(2), pp. 255–276.

Alzaatreh, A., Lee, C. and Famoye, F., (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), pp. 63–79.

Arellano-Valle, R. B.; Contreras-Reyes, J. E.; Stehlík, M., (2017). Generalized skew-normal negentropy and its application to fish condition factor time series. *Entropy*, 19, p. 528.

Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., Sharafi, F., et al., (2016). A new weighted lindley distribution with application. *Brazilian Journal of Probability and Statistics*, 30(1), pp. 1–27.

Asgharzadeh, A., Nadarajah, S. and Sharafi, F., (2018). Weibull lindley distribution. *REVSTAT–Statistical Journal*, 16(1), pp. 87–113.

Ashour, S. K., Eltehiwy, M. A., (2015). Exponentiated power lindley distribution. *Journal of advanced research*, 6(6), pp. 895–905.

Calabria, R., Pulcini, G., (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. *Communications in Statistics-Theory and Methods*, 25(3), pp. 585–600.

Contreras-Reyes, J. E., Kahrari, F. and Cortés, D. D., (2021). On the modified skew-normal-Cauchy distribution: Properties, inference and applications. *Communications in Statistics-Theory and Methods*, 50(15), pp. 3615-3631.

Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J. and Knuth, D. E., (1996). On the lambertw function. *Advances in Computational mathematics*, 5(1), pp. 329–359.

Galambos, J., Kotz, S., (2006). *Characterizations of Probability Distributions.: A Unified Approach with an Emphasis on Exponential and Related Models*, vol. 675. Springer.

Ghitany, M., Al-Mutairi, D. K., Balakrishnan, N. and Al-Enezi, L., (2013). Power lindley distribution and associated inference. *Computational Statistics Data Analysis*, 64, pp. 20–33.

Ghitany, M., Atieh, B. and Nadarajah, S., (2008). Lindley distribution and its application. *Mathematics and computers in simulation*, 78(4), pp. 493–506.

Gleaton, J. U., Lynch, J. D., (2010). Extended generalized log-logistic families of lifetime distributions with an application. *J. Probab. Stat. Sci*, 8, pp. 1–17.

Gomes-Silva, F. S., Percontini, A., de Brito, E., Ramos, M. W., Venancio, R., and Cordeiro, G. M., (2017). The odd lindley-g family of distributions. *Austrian Journal of Statistics*, 46(1), pp. 65–87.

Jodr, P., (2010). Computer generation of random variables with lindley or poisson–lindley distribution via the lambert w function. *Mathematics and Computers in Simulation*, 81(4), pp. 851–859.

Kim, J. H., Jeon, Y., (2013). Credibility theory based on trimming. *Insurance: Mathematics and Economics*, 53(1), pp. 36–47.

Kotz, S., Shanbhag, D., (1980). Some new approaches to probability distributions. *Advances in Applied Probability*, pp. 903–921.

Lindley, D. V., (1958). Fiducial distributions and bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, 20(1), pp. 102–107.

Marshall, A. W., Olkin, I., (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika*, 84(3), pp. 641–652.

Merovci, F., Sharma, V. K., (2014). The beta-lindley distribution: properties and applications. *Journal of Applied Mathematics*.

Murthy, D. P., Xie, M. and Jiang, R., (2004). *Weibull models*, vol. 505, John Wiley Sons.

Nadarajah, S., Bakouch, H. S. and Tahmasbi, R., (2011). A generalized lindley distribution. *Sankhya B*, 73(2), pp. 331–359.

Oluyede, B. O., Yang, T. and Makubate, B., (2016). A new class of generalized power lindley distribution with applications to lifetime data. *Asian Journal of Mathematics and Applications*, 2016, p.1.

Ozel, G., Alizadeh, M., Cakmakyapan, S., Hamedani, G., Ortega, E. M. and Cancho, V. G., (2017). The odd log- logistic lindley poisson model for lifetime data. *Communications in Statistics-Simulation and Computation*, 46(8), pp. 6513–6537.

Ranjbar, V., Alizadeh, M. and Altun, E., (2019). Extended generalized lindley distribution: Properties and applications. *Journal of Mathematical Extension*, 13(1), pp. 117–142.

Zakerzadeh, H., Dolati, A., (2009). Generalized lindley distribution. *Journal of Mathematical Extension*, 3(2), pp. 1–17.